A CLASS OF NON-DESARGUESIAN AFFINE PLANES[1]

BY

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I. **Introduction.** The only known finite non-Desarguesian projective or affine planes are the Veblen-Wedderburn planes, the dual Veblen-Wedderburn planes, and the Hughes planes [3].

In this paper, we give a construction for a class of affine planes. The collineation group for this class strongly resembles that of the Hughes planes. However, there are sufficient differences to insure that our planes are not, in general, Hughes planes.

II. **Algebraic preliminaries.** We shall be concerned with a finite left Veblen-Wedderburn system R—i.e., the left distributive law $a(b+c)=ab+ac$ holds in R. (For a complete set of postulates for a left Veblen-Wedderburn system, see [3].)

In addition, we require that R satisfy the following conditions: R is of order $n=q^2$ and contains a subfield F of order q and

(1) $ai+bi=(a+b)i$ if $i\in F$.

(2) $(ab)i=a(bi)$ if $i\in F$.

(3) The right distributive law does not hold in R.

It follows that R is a right vector space of dimension two over F.

Note that (1) follows immediately if each element of F commutes with every element of R and (1) implies (2) if q is a prime.

If q is a power of an odd prime, there is a left nearfield of order n satisfying these conditions. The construction given in [4, Theorem 6 and its Corollary] enables us to obtain Veblen-Wedderburn systems in which q is a prime and $GF(q)$ is in the center of R. These systems are not, in general, nearfields.

III. **Construction of planes.** Let R be a left Veblen-Wedderburn system satisfying the conditions (1), (2), (3) of part II. We define an affine plane as follows:

The points of Π are ordered pairs (x, y) of elements of R.

The lines of Π are the sets of points satisfying either one of the following two conditions:

(a) $y=xm+b$, m, b fixed, $m\notin F$.

(b) $x=ai+c$, $y=aj+d$, where $i, j\in F$ and $a\neq 0$, c, d are fixed elements of R.

For brevity, we shall denote the lines in case (b) by the symbol $\{ai+c, aj+d\}$. While we have not introduced a coordinate system in the

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usual sense, it will be convenient to refer to x and y as coordinates of the point (x, y).

Before proceeding to prove that Π is indeed an affine plane, let us examine the sets of points in case (b). If $\bar{\imath}$ is a fixed element of F, and if $\bar{a}=a\bar{\imath}$, then $\{ai+c, aj+d\}=\{\bar{a}i+c, \bar{a}j+d\}$. If $\bar{\imath}$ and $\bar{\jmath}$ are fixed elements of F and if $\bar{c}=c+a\bar{\imath}$, $\bar{d}=d+a\bar{\jmath}$, then $\{ai+c, aj+d\}=\{ai+\bar{c}, aj+\bar{d}\}$.

THEOREM 1. *The set of points and lines of* Π *constitute an affine plane.*

Proof. Consider the intersection of two lines of class (b), say $\{ai+c, aj+d\}$ and $\{\bar{a}i+\bar{c}, \bar{a}j+\bar{d}\}$. If (x, y) is in the intersection, then there exist $i, j, \bar{\imath}, \bar{\jmath}$ in F such that

$$x = ai + c = \bar{a}\bar{\imath} + \bar{c}, \qquad y = aj + d = \bar{a}\bar{\jmath} + \bar{d}.$$

Now the additive group of R is a right vector space of dimension two over F. If a and $\bar{a}$ are linearly dependent, we can, without loss of generality, take $a=\bar{a}$. In this case, either the two lines have no point in common (i.e., they are parallel) or

$$\bar{c} = c + a(i - \bar{\imath}), \qquad \bar{d} = d + a(j - \bar{\jmath})$$

and the two lines are identical.

On the other hand, if a and $\bar{a}$ are linearly independent, then $\bar{c}-c$ and $\bar{d}-d$ can each be uniquely represented in the form $ai-\bar{a}\bar{\imath}$ and $aj-\bar{a}\bar{\jmath}$, respectively, and the point (x, y) of intersection is uniquely determined.

For later purposes, we note that if $c=\bar{c}=0$, we must have $i=\bar{\imath}=0$. That is, two lines of the form $\{ai, aj+d\}$ intersect in a point for which the x coordinate is zero or are parallel.

It is well known in connection with the dual Veblen-Wedderburn planes that two lines of class (a) are parallel if they have the same value for m and otherwise have exactly one point in common.

We now consider the intersections of lines of type (a) with lines of type (b). Note that $\{ai+c, aj+d\}$ contains exactly $n=q^2$ points. For given $m, \bar{\imath}, \bar{\jmath}$ there exists a unique b such that $y=xm+b$ contains the point $(a\bar{\imath}+c, a\bar{\jmath}+d)$. Since the parameter b can assume exactly n different values, either

(1) For some b, $y=xm+b$ contains at least two points of $\{ai+c, aj+d\}$. Or

(2) Each line $y=xm+b$ contains exactly one point of $\{ai+c, aj+d\}$.

For given $m, b, \bar{\imath}, \bar{\jmath}, \bar{\imath}, \bar{\jmath}$ suppose that $(a\bar{\imath}+c, a\bar{\jmath}+d)$ and $(a\bar{\imath}+c, a\bar{\jmath}+d)$ are two distinct points on $y=xm+b$. Then

$$a\bar{\jmath} + d = (a\bar{\imath} + c)m + b, \qquad a\bar{\jmath} + d = (a\bar{\imath} + c)m + b$$

and

$$a(\bar{\jmath} - \bar{\jmath}) = (a\bar{\imath} + c)m - (a\bar{\imath} + c)m.$$

This last equation has a unique solution for m unless $\bar{\imath}=\bar{\imath}$. In this case, it follows that $\bar{\jmath}=\bar{\jmath}$, so that the two points are not distinct. If $\bar{\imath}\neq\bar{\imath}$, it is readily verified (using properties (1) and (2) of R as well as the left distributive law) that this equation is satisfied when $m=(\bar{\imath}-\bar{\imath})^{-1}(\bar{\jmath}-\bar{\jmath})\in F$.

It follows that, if $m \notin F$, $y = xm + b$ cannot contain two points of $\{ai+c,\ aj+d\}$. Hence each line of type (a) meets each line of type (b) exactly once.

We have established that lines belonging to different parallel classes have exactly one point in common.

Clearly each point belongs to exactly one line of each parallel class. Considering "equivalent" values of a, it follows that the number of parallel classes of type (b) is $(n-1)(q-1)^{-1}=q+1$. The number of values of $m\notin F$ is $n-q$. Hence, the total number of parallel classes is $n+1$. Since each line contains n points, it follows that the set of lines through a given point contains all of the points of Π. Thus every two points determine a line.

IV. **Collineations and comparisons with other planes.**

Definition. Let Π_0 denote the set of points for which the x coordinate is zero and lines of the form $\{ai,\ aj+b\}$.

Theorem 2. (1) *The mappings* $T(e)\colon (x, y)\to(x, y+e)$ *constitute a group of translations of* Π *with the points of* Π_0 *as a single transitive class.* (2) Π_0 *is an affine subplane of* Π.

Proof. (1) It is immediate that $T(e)$ carries $\{ai + c,\ aj + d\}$ into $\{ai+c,\ aj+d+e\}$ and the set for which $y=xm+b$ into the set for which $y=xm+b+e$. Since no affine point is fixed and each line is carried into a line parallel to it, the mapping is a translation. Moreover, the x coordinate is fixed, so that the image of each point in Π is in Π_0. Letting e take on all possible values, all of these points are in a single transitive class.

(2) Each line of the form $\{ai,\ aj+b\}$ contains q points of Π_0. We have already established that two lines of this kind which are not parallel intersect in a point of Π_0. It readily follows that Π_0 is an affine subplane of Π of order q.

Theorem 3. *The mappings*

$$M(\bar{\imath},\bar{\jmath})\colon (x, y) \to (x\bar{\imath},\ y\bar{\jmath}), \qquad \bar{\imath},\bar{\jmath} \neq 0, \in F,$$
$$E(\bar{\imath})\colon (x, y) \to (x,\ x\bar{\imath} + y), \qquad \bar{\imath} \in F,$$
$$A(\sigma)\colon (x, y) \to (x\sigma,\ y\sigma)$$

(*where* σ *is an automorphism of* R *which fixes all elements of* F) *are all collineations of* Π *which carry* Π_0 *into itself* (*provided that, in the case of* $M(\bar{\imath},\bar{\jmath})$, *the elements of* F *associate in the middle*).

Proof. $M(\bar{\imath},\bar{\jmath})$ carries $\{ai+c, aj+d\}$ into $\{ai+c\bar{\imath}, aj+d\bar{\jmath}\}$, $y=xm+b$ into $y=x(\bar{\imath}^{-1}m\bar{\jmath})+b\bar{\jmath}$.

$E(\bar{\imath})$ carries $\{ai+c,\ aj+d\}$ into $\{ai+c,\ aj+c\bar{\imath}+d\}$, $y=xm+b$ into $y=x(\bar{\imath}+m)+b$.

$A(\sigma)$ carries $\{ai+c,\ aj+d\}$ into $\{(a\sigma)i+c\sigma,\ (a\sigma)j+d\sigma\}$, $y=xm+b$ into $y=x(m\sigma)+b\sigma$.

Clearly, Π_0 is carried into itself in each case.

THEOREM 4. *If R is a nearfield, then $M(e)$: $(x, y)\to(ex, ey)$, $e\neq 0$, is a collineation of Π which carries Π_0 into itself. The points of Π then occur in at most two transitive classes under the full collineation group. These two classes consist respectively of the points which are and are not in Π_0.*

Proof. $M(e)$ carries $\{ai+c,\ aj+d\}$ into $\{eai+ec,\ eaj+ed\}$, $y=xm+b$ into $y=xm+eb$. Given $(x_1,\ y_1)$ and $(x_2,\ y_2)$, where $x_1x_2\neq 0$, the collineation $M(x_2x_1^{-1})T(y_2-x_2x_1^{-1}y_1)$ carries $(x_1,\ y_1)$ into $(x_2,\ y_2)$. The Theorem then follows from Theorem 2.

COROLLARY. *If R is a nearfield and $q\neq 3$, 5, 7, 11, 23, 59, then Π is not a Hughes plane.*

Proof. Since the Hughes planes are projective planes, we are really concerned with the projective plane Π^* obtained by adjoining the line at infinity to Π. The collineation group on the Hughes plane Λ divides the set of points in Λ into exactly two transitive classes: those in a fixed subplane Λ_0 and those which are not in Λ_0. The collineations fixing Λ_0 pointwise are automorphisms of a nearfield of order n [5; 6]. If $\Pi=\Lambda$, it follows from Theorem 4 that $\Pi_0=\Lambda_0$. But Π_0 is pointwise fixed by the group of collineations of order $q(q-1)$ generated by the $M(i,1)$ and $E(i)$. If $n=p^l=q^2$, the order of the group of automorphisms of a nearfield of order n must divide l, with the possible exceptions given in the Corollary [7]. Thus the group of collineations fixing Π_0 pointwise is larger than is the case in the Hughes plane and the Corollary follows.

LEMMA. *If Π admits a translation carrying $(0, 0)$ into $(c, 0)$, then $(e+c)m=em+cm$ for all e, m in R.*

Proof. Under the given translation, the line $\{ci,\ cj\}$ and all lines parallel to it must be fixed. The image of $\{ai,\ aj\}$ is $\{ai+c,\ aj\}$.

If $a\bar{i}=c\bar{i}+d$, $a\bar{j}=c\bar{j}+e$, then $\{ai,\ aj\}\cap\{ci+d,\ cj+e\}$ is $(a\bar{i},\ a\bar{j})$ and $\{ai+c,\ aj\}\cap\{ci+d,\ cj+e\}$ is $(a\bar{i}+c,\ a\bar{j})$. Hence each point of the form $(ai,\ aj)$ is mapped into $(ai+c,\ aj)$.

In particular, $(0,\ b)\to(c,\ b)$. Hence $\{ai,\ aj+b\}\to\{ai+c,\ aj+b\}$. By a process similar to the one above, we conclude that, in general, $(x, y)\to(x+c, y)$.

The line $y=xm+b$ must map into a line parallel to it through $(c,\ b)$, i.e., into $y=xm+b-cm$. Since $(e,\ em+b)$ is on $y=xm+b$, $(e+c,\ em+b)$ is on $y=xm+b-cm$. The Lemma then follows by substitution.

THEOREM 5. *The projective plane Π^* is not a Veblen-Wedderburn plane.*

Proof. A Veblen-Wedderburn plane admits all elations with some line l as axis. If the plane is finite and there is any collineation displacing l, the plane is Desarguesian and the corresponding affine planes are all translation planes.

It follows from the Lemma and condition (3) on R that Π is not a translation plane. The collineation group of Π^* displaces every line except the line at infinity of Π. The Theorem follows.

THEOREM 6. (1) *If R contains some fixed element $c \neq 0$ such that $(e+c)m = em+cm$ for all e, m in R, then the dual Veblen-Wedderburn plane coordinatised by R admits translations in all directions.*

(2) *If R contains no element c satisfying the above conditions, then* Π^* *is not a dual Veblen-Wedderburn plane.*

Proof. (1) The plane coordinatised by R admits all of the translations $(x, y) \rightarrow (x+c, y+b)$, where c is fixed and b may take on any value in R. (Note: This plane is not to be confused with Π.)

(2) A dual Veblen-Wedderburn plane has some point P which is fixed by all collineations (unless the plane is Desarguesian) and the plane admits all elations with P as center.

If Π^* is a dual Veblen-Wedderburn plane, P must be on the line at infinity, since there are no fixed points in Π. Moreover, the collineation group displaces every point on l_∞ corresponding to the lines $y = xm$, $m \in F$. Suppose that P is the point on l_∞ corresponding to $\{ci, cj\}$. Then Π^* admits all elations with center P and axis l. Thus Π admits a translation carrying (0, 0) into $(c, 0)$. The Theorem then follows from the Lemma.

REMARK. André [2] has given an example of an affine plane which is not a translation plane but admits translations in all directions. There are, however, no known finite planes having this property and no known finite left Veblen-Wedderburn systems satisfying the conditions of part (1) of Theorem 6.

We have not been able to determine whether or not Π^* admits any collineations which displace the line at infinity.

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